

# Symplectic realizations and symmetries of a Lotka-Volterra type system

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## Abstract

In this paper a Lotka-Volterra type system is considered. For such a system, bi-Hamiltonian formulation, symplectic realizations and symmetries are presented.<sup>1</sup>

**Keywords:** Lotka-Volterra system, symmetries, Hamiltonian dynamics, Lie groups

## 1 Introduction

The dynamical systems of Lotka-Volterra type have significant importance in biology for interaction models of biological species [19], [26], in chemistry for autocatalytic chemical reactions [7], [24], in hydrodynamics [5], [13], and so on.

These systems have been widely investigated from different points of view. We mention some studied topics, namely: integrals and invariant manifold [6], [16], [17], [18], Hamiltonian structure [15], [22], [25], [2], symmetries [1], [14], stability [17], [20], numerical integration [23], Kowalevski-Painleve property [8], and many others.

In our paper, the following Lotka-Volterra type system

$$\begin{cases} \dot{x} = x(by + cz) \\ \dot{y} = y(ax - by + cz + d) \\ \dot{z} = z(-ax + by - cz - d) \end{cases}, \quad (1.1)$$

where  $a, c \in \mathbf{R}^*$ ,  $b, d \in \mathbf{R}$ , is considered and some symmetries are given.

For our purposes, a Hamilton-Poisson realization and a symplectic realization of system (1.1) are required.

Theoretical details about symmetries of differential equations can be found in [4], [9], [11], [12], [21].

A similar study for Maxwell-Bloch equations was presented by P.A.Damianou and P.G.Paschali in [10].

## 2 Hamiltonian structures and symmetries for considered system in the case $d = 0$

In this section, we consider system (1.1) with  $d = 0$ , i.e.

$$\begin{cases} \dot{x} = x(by + cz) \\ \dot{y} = y(ax - by + cz) \\ \dot{z} = z(-ax + by - cz) \end{cases} \quad (2.1)$$

A bi-Hamiltonian structure, a symplectic realization and some symmetries of system (2.1) are given. For system (2.1), the functions  $H_1, H_2 \in \mathcal{C}^\infty(\mathbf{R}^3, \mathbf{R})$ ,

$$H_1(x, y, z) = yz$$

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and

$$H_2(x, y, z) = x(ax - 2by + 2cz)$$

are constants of motion.

Let us consider the linear Poisson algebra  $\mathcal{P}_1$ ,

$$\begin{aligned}\{x, y\}_1 &= \alpha_1 x + \alpha_2 y + \alpha_3 z, \\ \{x, z\}_1 &= \beta_1 x + \beta_2 y + \beta_3 z, \\ \{y, z\}_1 &= \gamma_1 x + \gamma_2 y + \gamma_3 z.\end{aligned}$$

Imposing the condition that  $C = H_1$  to be a Casimir for  $\mathcal{P}_1$ , it results  $\alpha_1 = \alpha_3 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = \gamma_3 = 0$ ,  $\beta_3 = -\alpha_2$ . If the Hamiltonian function is  $H = H_2$ , we get the following dynamical system:

$$\begin{cases} \dot{x} = -2\alpha_2 x(by + cz) \\ \dot{y} = -2\alpha_2 y(ax - by + cz) \\ \dot{z} = -2\alpha_2 z(-ax + by - cz) \end{cases} \quad (2.2)$$

Taking  $\alpha_2 = -\frac{1}{2}$ , the above system is the considered system (2.1). Thus,

$$\{x, y\}_1 = -\frac{1}{2}y, \quad \{x, z\}_1 = \frac{1}{2}z, \quad \{y, z\}_1 = 0,$$

or in coordinates, using matrix notation,

$$\pi_1(x, y, z) = \begin{bmatrix} 0 & -\frac{1}{2}y & \frac{1}{2}z \\ \frac{1}{2}y & 0 & 0 \\ -\frac{1}{2}z & 0 & 0 \end{bmatrix}.$$

Therefore we consider the three-dimensional Lie algebra  $g_1$  given by

$$[E_1, E_2] = -\frac{1}{2}E_2, [E_1, E_3] = \frac{1}{2}E_3, [E_2, E_3] = 0,$$

where

$$E_1 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

As a real vector space,  $g_1$  is generated by the base  $B_{g_1} = \{E_1, E_2, E_3\}$ , whence

$$g_1 = \{X \in gl(3, \mathbf{R}) \mid X = \begin{bmatrix} -\alpha & 0 & \beta \\ 0 & \alpha & \gamma \\ 0 & 0 & 0 \end{bmatrix}, \alpha, \beta, \gamma \in \mathbf{R}\}.$$

In order to provide the Lie group  $G_1$  generated by the Lie algebra  $g_1$ , we consider

$$A = \exp(kE_1) \cdot \exp(lE_2) \cdot \exp(mE_3) = \begin{bmatrix} e^{-\frac{1}{2}k} & 0 & le^{-\frac{1}{2}k} \\ 0 & e^{\frac{1}{2}k} & me^{\frac{1}{2}k} \\ 0 & 0 & 1 \end{bmatrix}$$

Taking  $k = 2u$ ,  $l = ve^u$ ,  $m = we^{-u}$ , it follows

$$G_1 = \{A \in GL(3, \mathbf{R}) \mid A = \begin{bmatrix} e^{-u} & 0 & v \\ 0 & e^u & w \\ 0 & 0 & 1 \end{bmatrix}, u, v, w \in \mathbf{R}\}.$$

In the same manner, interchanging  $H_1$  to  $H_2$ , we get the linear Poisson algebra  $\mathcal{P}_2$ :

$$\{x, y\}_2 = cx, \{x, z\}_2 = bx, \{y, z\}_2 = ax - by + cz,$$

or

$$\pi_2(x, y, z) = \begin{bmatrix} 0 & cx & bx \\ -cx & 0 & ax - by + cz \\ -bx & -ax + by - cz & 0 \end{bmatrix}.$$

Let us consider the 3D Lie algebra  $g_2$ , given by

$$\begin{aligned} [X_1, X_2] &= cX_1, [X_1, X_3] = bX_1, \\ [X_2, X_3] &= aX_1 - bX_2 + cX_3, \end{aligned}$$

where

$$X_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, X_2 = \begin{bmatrix} -c & \frac{a}{b} & c \\ 0 & 0 & 0 \\ 0 & \frac{a}{b} & c \end{bmatrix}, X_3 = \begin{bmatrix} -b & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & b \end{bmatrix},$$

$b \neq 0$ .

The Lie group  $G_2$  generated by the Lie algebra  $g_2$  has the generic element

$$\begin{aligned} A &= \exp(\alpha X_1) \cdot \exp(\beta X_2) \cdot \exp(\gamma X_3) = \\ &= \begin{bmatrix} e^{-\beta c - \gamma b} & \alpha + \frac{a}{2bc}(e^{\beta c} - e^{-\beta c}) & \frac{1}{2}(e^{\beta c + \gamma b} - e^{-\beta c - \gamma b}) \\ 0 & 1 & 0 \\ 0 & \frac{a}{bc}(e^{\beta c} - 1) & e^{\beta c + \gamma b} \end{bmatrix} \end{aligned}$$

Taking  $\alpha = u, e^\beta = v, e^\gamma = w$ , we obtain

$$G_2 = \left\{ A = \begin{bmatrix} v^{-c}w^{-b} & u + \frac{a}{2bc}(v^c - v^{-c}) & \frac{1}{2}(v^c w^b - v^{-c}w^{-b}) \\ 0 & 1 & 0 \\ 0 & \frac{a}{bc}(v^c - 1) & v^c w^b \end{bmatrix}, u, v, w \in \mathbf{R}, v > 0, w > 0 \right\}.$$

In the case  $b = 0$ , the base  $\{Y_1, Y_2, Y_3\}$  of  $g_2$  is given by

$$Y_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Y_2 = \begin{bmatrix} c & a & -c \\ 0 & 2c & 0 \\ 0 & a & 0 \end{bmatrix}, Y_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the corresponding Lie group is

$$G_2^{b=0} = \left\{ A = \begin{bmatrix} e^{\beta c} & \frac{a}{2c}(e^{2\beta c} - 1) + \alpha e^{2\beta c} & 1 - (1 + \gamma)e^{\beta c} \\ 0 & e^{2\beta c} & 0 \\ 0 & \frac{a}{2c}(e^{2\beta c} - 1) & 1 \end{bmatrix}, \alpha, \beta, \gamma \in \mathbf{R} \right\}.$$

Since

$$\pi_1 \cdot \nabla H_2 = \pi_2 \cdot \nabla H_1 = \begin{pmatrix} x(by + cz) \\ y(ax - by + cz) \\ z(-ax + by - cz) \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix},$$

system (2.1) is a bi-Hamiltonian system. For the  $\pi_1$  bracket,  $H_2$  is the Hamiltonian and  $H_1$  is a Casimir. For the  $\pi_2$  bracket,  $H_1$  is the Hamiltonian and  $H_2$  is a Casimir.

We recall that for a system  $\dot{x} = f(x)$ , where  $f : M \rightarrow TM$ , and  $M$  is a smooth manifold of finite dimension, a vector field  $\mathbf{X}$  is called:

- a *symmetry* if  $\frac{\partial \mathbf{X}}{\partial t} + [\mathbf{X}, \mathbf{X}_f] = 0$ , where  $\mathbf{X}_f$  is the vector field defined by the system;
- a *Lie-point symmetry* if its first prolongation transforms solutions of the system into other solutions;

- *a conformal symmetry* if the Lie derivative along  $\mathbf{X}$  satisfies  $L_{\mathbf{X}}\pi = \lambda\pi$  and  $L_{\mathbf{X}}H = \nu H$ , for some scalars  $\lambda, \nu$ , where the Poisson tensor  $\pi$  and the Hamiltonian  $H$  give the Hamilton-Poisson realization of the system;

- *a master symmetry* if  $[[\mathbf{X}, \mathbf{X}_f], \mathbf{X}_f] = 0$ , but  $[\mathbf{X}, \mathbf{X}_f] \neq 0$ .

First, we give a characterization of the vector field

$$\mathbf{X} = \alpha t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}, \quad \alpha \in \mathbf{R},$$

to be a Lie-point symmetry for system

$$\begin{cases} \dot{x} = f(x, y, z) \\ \dot{y} = g(x, y, z) \\ \dot{z} = h(x, y, z) \end{cases} \quad (2.3)$$

**Theorem 2.1.** *Let  $f, g, h$  be three real functions of class  $C^1$  on a cone in  $\mathbf{R}^3$ .*

*The vector field*

$$\mathbf{X} = \alpha t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}, \quad \alpha \in \mathbf{R},$$

*is a Lie-point symmetry of system (2.3) if and only if  $f, g, h$  are homogeneous functions of degree  $1 - \alpha$ .*

**Proof.** The vector field  $\mathbf{X}$  is a Lie point symmetry for system (2.3) if and only if its first prolongation

$$pr^{(1)}(\mathbf{X}) = \mathbf{X} + (\dot{x} - \alpha x) \frac{\partial}{\partial \dot{x}} + (\dot{y} - \alpha y) \frac{\partial}{\partial \dot{y}} + (\dot{z} - \alpha z) \frac{\partial}{\partial \dot{z}}$$

applied to the system equations vanishes. This condition is equivalent to

$$\begin{cases} (1 - \alpha)f(x, y, z) - x \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial y} - z \frac{\partial f}{\partial z} = 0 \\ (1 - \alpha)g(x, y, z) - x \frac{\partial g}{\partial x} - y \frac{\partial g}{\partial y} - z \frac{\partial g}{\partial z} = 0 \\ (1 - \alpha)h(x, y, z) - x \frac{\partial h}{\partial x} - y \frac{\partial h}{\partial y} - z \frac{\partial h}{\partial z} = 0 \end{cases},$$

that is  $f, g, h$  are homogeneous functions of degree  $1 - \alpha$ .

The next result one furnishes a Lie point symmetry of system (2.1) and a conformal symmetry.

**Proposition 2.2.** *The vector field*

$$\mathbf{X} = -t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$$

*is a Lie point symmetry of system (2.1). Moreover,  $\mathbf{X}$  is a conformal symmetry.*

**Proof.** Using **Theorem 2.1**, it follows that  $\mathbf{X}$  is a Lie point symmetry of system (2.1).

One can easily check that

$$L_{\mathbf{X}}\pi_1 = -\pi_1, L_{\mathbf{X}}\pi_2 = -\pi_2, L_{\mathbf{X}}H_1 = 2H_1, L_{\mathbf{X}}H_2 = 2H_2,$$

whence  $\mathbf{X}$  is a conformal symmetry.

The following result provides a master symmetry of our considered system.

**Proposition 2.3.** *The vector field*

$$\begin{aligned} \vec{X} &= (k_1x + k_2bxy + k_2cxz) \frac{\partial}{\partial x} + \\ &+ (k_1y + k_2axy - k_2by^2 + k_2cyz) \frac{\partial}{\partial y} + \\ &+ (k_1z - k_2axz + k_2byz - k_2cz^2) \frac{\partial}{\partial z}, \end{aligned}$$

*where  $k_1 \in \mathbf{R}^*$ ,  $k_2 \in \mathbf{R}$ , is a master symmetry of system (2.1).*

**Proof.** We denote by  $\vec{V}$  the associated vector field of system (2.1), that is

$$\vec{V} = (bxy + cxz) \frac{\partial}{\partial x} + (axy - by^2 + cyz) \frac{\partial}{\partial y} + (-axz + byz - cz^2) \frac{\partial}{\partial z}.$$

It follows that the following relations

$$[\vec{X}, \vec{V}] = k_1 \vec{V}, \quad [[\vec{X}, \vec{V}], \vec{V}] = \vec{0}$$

hold.

Therefore  $\vec{X}$  is a master symmetry of system (2.1).

In the following a symplectic realization of system (2.1) is given. Using this fact, the symmetries of Newton's equations are studied.

The next theorem states that the system (2.1) can be regarded as a Hamiltonian mechanical system.

**Theorem 2.4.** *The Hamilton-Poisson mechanical system  $(\mathbf{R}^3, \pi_1, H_2)$  has a full symplectic realization  $(\mathbf{R}^4, \omega, \tilde{H})$ , where  $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$  and*

$$\tilde{H} = \frac{1}{a} \left[ p_1^2 - (bp_2 e^{\frac{a}{2} q_1} - ce^{-\frac{a}{2} q_1})^2 \right].$$

**Proof.** The corresponding Hamilton's equations are

$$\begin{cases} \dot{q}_1 = \frac{2}{a} p_1 \\ \dot{q}_2 = -\frac{2b^2}{a} p_2 e^{aq_1} + \frac{2bc}{a} \\ \dot{p}_1 = -c^2 e^{-aq_1} + b^2 p_2^2 e^{aq_1} \\ \dot{p}_2 = 0 \end{cases} \quad (2.4)$$

We define the application  $\varphi : \mathbf{R}^4 \rightarrow \mathbf{R}^3$  by

$$\varphi(q_1, q_2, p_1, p_2) = (x, y, z),$$

where

$$\begin{aligned} x &= \frac{1}{a} p_1 + \frac{b}{a} p_2 e^{\frac{a}{2} q_1} - \frac{c}{a} e^{-\frac{a}{2} q_1} \\ y &= p_2 e^{\frac{a}{2} q_1} \\ z &= e^{-\frac{a}{2} q_1}. \end{aligned}$$

It follows that  $\varphi$  is a surjective submersion, the equations (2.4) are mapped onto the equations (2.1), the canonical structure  $\{.,.\}_\omega$  is mapped onto the Poisson structure  $\pi_1$ , as required.

We also remark that  $H_2 = \tilde{H}$  and  $H_1 = p_2$ .

It is natural to ask which non-canonical bracket  $\tilde{\omega}$  in  $\mathbf{R}^4$  is mapped onto the  $\pi_2$  bracket.

Considering now the Hamiltonian  $H = p_2$  and taking into account the Jacobi's identity, it results

$$\begin{aligned} \{q_1, p_2\}_{\tilde{\omega}} &= \frac{2}{a} p_1, \\ \{q_2, p_2\}_{\tilde{\omega}} &= \frac{2b}{a} (c - bp_2 e^{aq_1}), \\ \{p_1, p_2\}_{\tilde{\omega}} &= b^2 p_2^2 e^{aq_1} - c^2 e^{-aq_1}, \\ \{q_1, p_1\}_{\tilde{\omega}} &= \frac{2b}{a} (bp_2 e^{aq_1} - c), \\ \{q_1, q_2\}_{\tilde{\omega}} &= F(p_1, p_2, q_1, q_2), \\ \{p_1, q_2\}_{\tilde{\omega}} &= G(p_1, p_2, q_1, q_2), \end{aligned}$$

where the functions  $F$  and  $G$  satisfies the relations:

$$\begin{aligned}
& -\frac{2}{a}p_1\frac{\partial F}{\partial q_1} + \frac{2b}{a}(bp_2e^{aq_1} - c)\frac{\partial F}{\partial q_2} + (c^2e^{-aq_1} - b^2p_2^2e^{aq_1})\frac{\partial F}{\partial p_1} + \frac{2}{a}G = \frac{4b^2}{a^2}p_1e^{aq_1} \\
& \frac{2}{a}p_1\frac{\partial G}{\partial q_1} - \frac{2b}{a}(bp_2e^{aq_1} - c)\frac{\partial G}{\partial q_2} - (c^2e^{-aq_1} - b^2p_2^2e^{aq_1})\frac{\partial G}{\partial p_1} - a(b^2p_2^2e^{aq_1} + c^2e^{-aq_1})F = \\
& = \frac{6b^4}{a}p_2^2e^{2aq_1} - \frac{8b^3c}{a}p_2e^{aq_1} + \frac{2b^2c^2}{a} \\
& \frac{2b(c - bp_2e^{aq_1})}{a}\left(\frac{\partial F}{\partial q_1} + \frac{\partial G}{\partial p_1}\right) + G\frac{\partial F}{\partial q_2} - (c^2e^{-aq_1} - b^2p_2^2e^{aq_1})\frac{\partial F}{\partial p_2} - F\frac{\partial G}{\partial q_2} - \frac{2p_1}{a}\frac{\partial G}{\partial p_2} + 2b^2p_2e^{aq_1}F = \\
& = \frac{4b^3}{a^2}(ce^{aq_1} - bp_2e^{2aq_1}).
\end{aligned}$$

In the particular case  $b = 0$  one finds

$$F_1(p_1, p_2, q_1, q_2) = p_1, \quad G_1(p_1, p_2, q_1, q_2) = -\frac{ac^2}{2}e^{-aq_1}$$

and respectively

$$\begin{aligned}
F_2(p_1, p_2, q_1, q_2) &= p_1 + e^{-aq_1}, \\
G_2(p_1, p_2, q_1, q_2) &= -\frac{a}{2}e^{-aq_1}(2p_1 + c^2).
\end{aligned}$$

Thus, we obtain two non-canonical brackets  $\tilde{\omega}_1$  and  $\tilde{\omega}_2$ .

The first one is compatible with  $\omega$  but it is degenerate.

A symmetry of system (2.4) is

$$Z_0 = -t\frac{\partial}{\partial t} + p_1\frac{\partial}{\partial p_1} + 2p_2\frac{\partial}{\partial p_2} - \frac{2}{a}\frac{\partial}{\partial q_1} + \left(\frac{3a}{2}p_2 - q_2\right)\frac{\partial}{\partial q_2}$$

which satisfies

$$L_{Z_0}(J_0) = -J_0, \quad L_{Z_0}(J_1) = -J_1, \quad L_{Z_0}(\tilde{H}) = 2\tilde{H},$$

where  $J_0$  and  $J_1$  are respectively Poisson tensors associated with  $\omega$  and  $\tilde{\omega}_1$ .

Defining the operator  $\mathcal{R} = J_1J_0^{-1}$ , we get a symmetry

$$Z_1 = \mathcal{R}Z_0 = \frac{1}{2}c^2e^{-aq_1}(2q_2 - ap_2)\frac{\partial}{\partial p_1} + \frac{2}{a}(p_1^2 - c^2e^{-aq_1})\frac{\partial}{\partial p_2} + \frac{p_1}{a}(ap_2 - 2q_2)\frac{\partial}{\partial q_1} + (p_1^2 - c^2e^{-aq_1})\frac{\partial}{\partial q_2}$$

of system (2.4). This symmetry leads to a symmetry of system (2.1) for  $b = 0$ :

$$\vec{Z} = \frac{c}{4}xz(2k - ayz)\frac{\partial}{\partial x} - \frac{1}{4}(4a\frac{x^2}{z} + 8cx + a^2xy^2z + acy^2z^2 - 2akxy - 2ckyz)\frac{\partial}{\partial y} - \frac{z}{4}(ax + cz)(2k - ayz)\frac{\partial}{\partial z},$$

$k \in \mathbf{R}$ , which sending  $\pi_1$  bracket to  $\pi_2$  and  $H_1$  to  $H_2$ . It follows that  $\vec{Y} = \vec{Z} + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$  is a master symmetry of system (2.1) which sending  $\pi_1$  bracket to  $\pi_2 - \pi_1$  and  $H_1$  to  $2H_1 - H_2$ .

The second bracket  $\tilde{\omega}_2$  is non-degenerate but it is not compatible with  $\omega$  and they do not generate a recursion operator.

In the sequel we find the symmetries of Newton's equations. In the case  $b \neq 0$ , from Hamilton's equations (2.4) one obtains Newton's equations:

$$\ddot{q}_1 - \frac{a}{2b^2}e^{-aq_1}\dot{q}_2^2 + \frac{2c}{b}e^{-aq_1}\dot{q}_2 = 0 \quad (2.5)$$

$$\ddot{q}_2 - a\dot{q}_1\dot{q}_2 + 2bc\dot{q}_1 = 0. \quad (2.6)$$

These are also Lagrange's equations generated by the Lagrangian

$$L = \frac{a}{4}\dot{q}_1^2 + \left(\frac{c}{b}\dot{q}_2 - \frac{a}{4b^2}\dot{q}_2^2\right)e^{-aq_1}.$$

A vector field

$$\vec{v} = \xi(q_1, q_2, t) \frac{\partial}{\partial t} + \eta_1(q_1, q_2, t) \frac{\partial}{\partial q_1} + \eta_2(q_1, q_2, t) \frac{\partial}{\partial q_2}$$

is a Lie-point symmetry for Newton's equations if the action of its second prolongation on Newton's equations vanishes. Thus, for the equation (2.6), the following condition is obtained:

$$\begin{aligned} & (-2bc\eta_{1,t} - \eta_{2,tt}) + \dot{q}_1(-2bc\eta_{1,q_1} - 2bc\xi_t + a\eta_{2,t} - 2\eta_{2,tq_1} + 2bc\eta_{2,q_2}) + \dot{q}_1^2(-2bc\xi_{q_1} + a\eta_{2,q_1} - \eta_{2,q_1q_1}) + \\ & + \dot{q}_2 \left( a\eta_{1,t} - 2bc\eta_{1,q_2} - 2\eta_{2,tq_2} + \frac{2c}{b}e^{-aq_1}\eta_{2,q_1} + \xi_{tt} \right) + \\ & + \dot{q}_2^2 \left( a\eta_{1,q_2} - \eta_{2,q_2q_2} - \frac{a}{2b^2}e^{-aq_1}\eta_{2,q_1} + 2\xi_{tq_2} - \frac{2c}{b}e^{-aq_1}\xi_{q_1} \right) + \dot{q}_2^3 \left( \xi_{q_2q_2} + \frac{a}{2b^2}e^{-aq_1}\xi_{q_1} \right) + \\ & + \dot{q}_1\dot{q}_2(a\eta_{1,q_1} - 2\eta_{2,q_1q_2} + 2\xi_{tq_1} - 4bc\xi_{q_2}) + \dot{q}_1^2\dot{q}_2\xi_{q_1q_1} + \dot{q}_1\dot{q}_2^2(2\xi_{q_1q_2} + a\xi_{q_2}) = 0. \end{aligned}$$

The above equation must be satisfied identically in  $t, q_1, q_2, \dot{q}_1, \dot{q}_2$ , that are all independent. Doing standard manipulation, we obtain:

$$\begin{aligned} \eta_2 &= \eta_2(q_2, t), \quad \eta_1 = -\frac{1}{2bc} \cdot \frac{\partial \eta_2}{\partial t} + k_1 \\ \xi &= \xi(t), \quad \xi'(t) = \frac{a}{2bc} \cdot \frac{\partial \eta_2}{\partial t} + \frac{\partial \eta_2}{\partial q_2}, \end{aligned}$$

where  $k_1$  is an arbitrary real constant.

Taking into account the above result, the second prolongation of  $\vec{v}$  on equation (2.5) gives us:

$$\begin{aligned} & \dot{q}_2^2 \left( -\frac{a^2}{2b^2}e^{-aq_1}\eta_1 + \frac{a}{b^2}e^{-aq_1}\eta_{2,q_2} - \eta_{1,q_2q_2} \right) - \dot{q}_1\dot{q}_2a\eta_{1,q_2} + \dot{q}_1(\xi_{tt} + 2bc\eta_{1,q_2}) + \\ & + \dot{q}_2 \left( \frac{2ac}{b}e^{-aq_1}\eta_1 + \frac{a}{b^2}e^{-aq_1}\eta_{2,t} - \frac{2c}{b}e^{-aq_1}\eta_{2,q_2} - \frac{2c}{b}e^{-aq_1}\xi_t - 2\eta_{1,tq_2} \right) + \left( -\eta_{1,tt} - \frac{2c}{b}e^{-aq_1}\eta_{2,t} \right) = 0. \end{aligned}$$

We get the overall result:

$$\begin{cases} \xi = \alpha t + \beta \\ \eta_1 = 2\alpha \\ \eta_2 = a\alpha q_2 + \gamma \end{cases},$$

where  $\alpha, \beta, \gamma$  are real constants.

Now, we conclude the following result:

**Theorem 2.5.** *The symmetries of Newton's equations are given by*

$$\vec{v} = (\alpha t + \beta) \frac{\partial}{\partial t} + 2\alpha \frac{\partial}{\partial q_1} + (a\alpha q_2 + \gamma) \frac{\partial}{\partial q_2},$$

where  $\alpha, \beta, \gamma \in \mathbf{R}$ .

**Remark 2.1.** (i) For  $\alpha = \gamma = 0$  and  $\beta \neq 0$ , we have  $\vec{v}_1 = \beta \frac{\partial}{\partial t}$  that represents the time translation symmetry which generates the conservation of energy  $H$ .

(ii) For  $\alpha = \beta = 0$  and  $\gamma \neq 0$ , we have  $\vec{v}_2 = \gamma \frac{\partial}{\partial q_2}$  that represents a translation in the cyclic  $q_2$  direction which is related to the conservation of  $p_2$ .

Moreover, using the Lagrangian  $L$  and Noether's theory we deduce that both  $\vec{v}_1$  and  $\vec{v}_2$  are variational symmetries since they satisfy the condition  $pr^{(1)}(\vec{v})L + L\text{div}(\xi) = 0$ .

**Remark 2.2.** The 3-dimensional Lie algebra corresponding to the symmetries of Newton's equations endowed with the standard Lie bracket vector fields is generated by the base  $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ , where

$$\begin{aligned} \vec{u}_1 &= -t \cdot \frac{\partial}{\partial t} - \frac{2}{a} \cdot \frac{\partial}{\partial q_1} - q_2 \cdot \frac{\partial}{\partial q_2} \\ \vec{u}_2 &= \frac{\partial}{\partial t} \\ \vec{u}_3 &= \frac{\partial}{\partial q_2}. \end{aligned}$$

The following relations

$$[\vec{u}_1, \vec{u}_2] = \vec{u}_2, \quad [\vec{u}_1, \vec{u}_3] = \vec{u}_3, \quad [\vec{u}_2, \vec{u}_3] = \vec{0}$$

hold. Therefore this Lie algebra is of type V in Bianchi classification [3].

### 3 Hamiltonian structures and symmetries for considered system in the case $d \neq 0$

Let us consider system (1.1) in the case  $d \neq 0$ .

In this section a symplectic realization of system (1.1) is given. Using this fact, the symmetries of Newton's equations are studied.

For our purpose we can use the same Hamilton-Poisson realization  $(\mathbf{R}^3, \pi_1, H_2)$  as like as in the case  $d = 0$ , but for the sake of simplicity we choose another realization. We consider the following Lie-Poisson structure:

$$\pi(x, y, z) = \begin{bmatrix} 0 & \frac{c}{ad}(ax - 2by) & \frac{b}{ad}(ax + 2cz) \\ -\frac{c}{ad}(ax - 2by) & 0 & \frac{1}{d}(ax - by + cz + d) \\ -\frac{b}{ad}(ax + 2cz) & -\frac{1}{d}(ax - by + cz + d) & 0 \end{bmatrix}.$$

The Hamiltonian  $H$  is given by

$$H(x, y, z) = dyz,$$

and moreover, the function  $C$ ,

$$C(x, y, z) = x(ax - 2by + 2cz + 2d) - \frac{4bc}{a}yz,$$

is a Casimir of our configuration.

The next theorem states that the system (1.1) can be regarded as a Hamiltonian mechanical system.

**Theorem 3.1.** *The Hamilton-Poisson mechanical system  $(\mathbf{R}^3, \pi, H)$  has a full symplectic realization*

$$(\mathbf{R}^4, \omega, \tilde{H}),$$

where

$$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$$

and

$$\tilde{H} = bcdq_1^2 - \frac{1}{16bcd} (ap_2 - p_1^2 + 4b^2c^2q_1^2)^2.$$

**Proof.** The corresponding Hamilton's equations are

$$\begin{cases} \dot{q}_1 = \frac{1}{4bcd} p_1 (ap_2 - p_1^2 + 4b^2c^2q_1^2) \\ \dot{q}_2 = -\frac{a}{8bcd} (ap_2 - p_1^2 + 4b^2c^2q_1^2) \\ \dot{p}_1 = -2bcdq_1 + \frac{bc}{d} (ap_2 - p_1^2 + 4b^2c^2q_1^2) q_1 \\ \dot{p}_2 = 0 \end{cases} \quad (3.1)$$

We define the application  $\varphi : \mathbf{R}^4 \rightarrow \mathbf{R}^3$  by

$$\varphi(q_1, q_2, p_1, p_2) = (x, y, z),$$



where

$$\begin{aligned}x &= \frac{1}{a}p_1 + \frac{1}{2ad}(ap_2 - p_1^2 + 4b^2c^2q_1^2) - \frac{d}{a} \\y &= cq_1 + \frac{1}{4bd}(ap_2 - p_1^2 + 4b^2c^2q_1^2) \\z &= bq_1 - \frac{1}{4cd}(ap_2 - p_1^2 + 4b^2c^2q_1^2)\end{aligned}$$

It follows that  $\varphi$  is a surjective submersion, the equations (3.1) are mapped onto the equations (1.1), the canonical structure  $\{.,.\}_\omega$  is mapped onto the Poisson structure  $\pi$ , as required.

We also remark that  $H = \tilde{H}$  and  $C = p_2$ .

From Hamilton's equations (3.1) we obtain by differentiation, Newton's equations:

$$\begin{aligned}4\ddot{q}_2\dot{q}_2 - a^2q_1\dot{q}_1 &= 0 \\4a^2\ddot{q}_1\dot{q}_2^2 - 64b^2c^2\dot{q}_2^4q_1 - 16abcd\dot{q}_2^3q_1 - a^4\dot{q}_1^2q_1 &= 0\end{aligned}$$

These are also Lagrange's equations generated by the Lagrangian

$$L = \frac{a}{4}\frac{\dot{q}_1^2}{\dot{q}_2} + \frac{4bcd}{a^2}\dot{q}_2^2 + \frac{4b^2c^2}{a}\dot{q}_2q_1^2 + bcdq_1^2.$$

The condition for the vector field

$$\vec{v} = \xi(q_1, q_2, t)\frac{\partial}{\partial t} + \eta_1(q_1, q_2, t)\frac{\partial}{\partial q_1} + \eta_2(q_1, q_2, t)\frac{\partial}{\partial q_2}$$

to be a Lie Point symmetry for Newton's equations leads to:  $4\dot{q}_2\ddot{\eta}_2 - 4\dot{q}_2^2\ddot{\xi} - a^2q_1\dot{\eta}_1 + 4\ddot{q}_2\dot{\eta}_2 + \dot{\xi}(a^2q_1\dot{q}_1 - 12\ddot{q}_2\dot{q}_2) - a^2\dot{q}_1\eta_1 = 0$

$4a^2\dot{q}_2^2\dot{\eta}_1 - 4a^2\dot{q}_1\dot{q}_2^2\ddot{\xi} - 2a^4\dot{q}_1q_1\dot{\eta}_1 + (8a^2\ddot{q}_1\dot{q}_2 - 256b^2c^2\dot{q}_2^3q_1 - 48abcd\dot{q}_2^2q_1)\dot{\eta}_2 + (2a^4\dot{q}_1^2q_1 - 16a^2\ddot{q}_1\dot{q}_2^2 + 256b^2c^2\dot{q}_2^4q_1 + 48abcd\dot{q}_2^3q_1)\dot{\xi} - (64b^2c^2\dot{q}_2^4 + 16abcd\dot{q}_2^3 + a^4\dot{q}_1^2)\eta_1 = 0.$

The resulting equations obtained by expanding  $\dot{\xi}, \ddot{\xi}, \dot{\eta}_1, \ddot{\eta}_1, \dot{\eta}_2, \ddot{\eta}_2$  and replacing  $\ddot{q}_1$  and  $\ddot{q}_2$  must be satisfied identically in  $t, q_1, q_2, \dot{q}_1, \dot{q}_2$ , that are all independent. Doing standard manipulation, we get the overall result:

$$\begin{cases} \xi = k_1 \\ \eta_1 = 0 \\ \eta_2 = k_2 \end{cases}$$

where  $k_1, k_2$  are real constants.

For  $k_2 = 0$  and  $k_1 \neq 0$ , we have  $\vec{v}_1 = k_1\frac{\partial}{\partial t}$  that represents the time translation symmetry which generates the conservation of energy  $\tilde{H}$ .

For  $k_1 = 0$  and  $k_2 \neq 0$ , we have  $\vec{v}_2 = k_2\frac{\partial}{\partial q_2}$  that represents a translation in the cyclic  $q_2$  direction which is related to the conservation of  $p_2$ .

Moreover, using the Lagrangian  $L$  and Noether's theory we deduce that both  $\vec{v}_1$  and  $\vec{v}_2$  are variational symmetries since they satisfy the condition  $pr^{(1)}(\vec{v})L + L\text{div}(\xi) = 0$ .

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